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Bang-Bang Free Boundary Control of a Stefan Problem for Metallurgical Length Maintenance

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***Abstract*— In continuous casting of steel, metallurgical length (ML) is the distance between the exit from the mold and the point of full solidification of a steel slab. This work explores the potential of using the open-loop spray-cooling control to minimize ML deviations from the desired location during casting speed changes under spray flow rate constraints. This objective essentially reduces to motion planning, i.e. apriori generation of spray flow rate commands, which when applied to the process make the latter execute the motion that reduces ML deviations from the setpoint in the shortest time possible. The existence and uniqueness of the solution of the single-phase one-dimensional (1D) Stefan solidification model and its two-dimensional (2D) extension representing the solidifying slab cross-section under bounded bang-bang control and some simplifying but practically justified assumptions are proved. The general synthesis setting for bang-bang control of the single-phase 1D Stefan problem and its 2D extension under boundary flux input constraints is formulated. Then, the bang-bang control for the minimization of the ML deviation from the desired value after the casting speed increase is heuristically found for the 2D slab model through trial-and-error. The simulation results of bang-bang ML control are provided.**

I. Introduction

Processes involving solidification are wide-spread in manufacturing but pose several significant challenges to traditional control theoretic methods due to being fundamentally infinite-dimensional and nonlinear in nature. The simplest, but still accurate, model of such processes, commonly called the Stefan Problem, splits the spatial domain into separate sub-domains for the liquid and the solid parts of the material. Within the sub-domains, temperature follows the usual parabolic heat-diffusion partial differential equation (PDE). The boundary between the domains moves according to the conservation of energy, written as the Stefan condition in terms of the temperature gradients on both sides of the boundary. Consider a special solidification process: continuous casting, which as of 2016 was used to make more than 96.2% of the steel in the world [1]. An illustration of this process is shown in Figure 1. A typical continuous casting operation keeps a constant flow rate of liquid metal into the machine. The metal in the caster, called the strand, cools and solidifies as it moves through the machine. Steel solidification starts in the mold – the primary cooling zone. Below the mold is the area referred to as the secondary cooling zone where heat is removed by water cooling sprays and support rolls through direct contact with the strand surface. At the exit of the caster, the fully solid metal is cut into separate pieces either to be processed further or shipped directly to customer

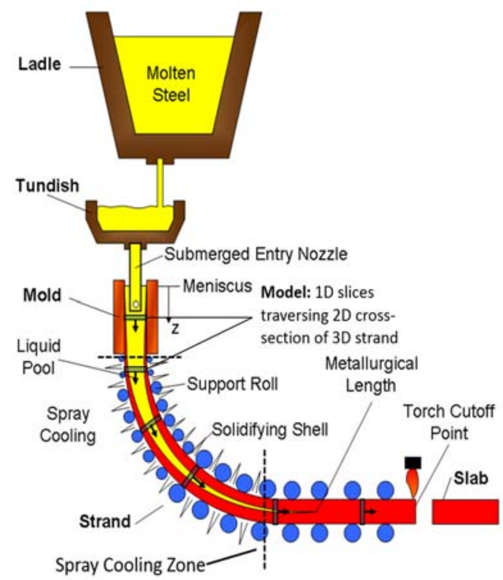

Figure 1. Illustration of continuous steel slab caster.

In continuous casting of steel, strand surface temperature and metallurgical length (ML) - the distance between the meniscus and the point of full steel slab solidification - are two key processing variables that require real-time control to meet product quality and operational safety demands. The curved strand of steel undergoes unbending, which causes tensile stress on the inside radius surface that results in transverse cracks if the steel is too brittle. The focus of the control methods currently used in the steel industry is to optimize the surface temperature profile down the caster to avoid stress, like that arising due to unbending, in the temperature regions of low ductility to minimize the possible creation of cracks. However, ML regulation may be more important for operations limited by the casting speed, or for steel grades that are more sensitive to centerline defects, such as centerline segregation, than to surface defects. Centerline segregation is a type of macro-segregation that appears as a line of impurities distributed near the centerline along the slab length. It is usually associated with other defects, such as centerline porosity, inclusions, alloy-rich regions, cracks, and other undesired property variations. These centerline defects are often very harmful, especially in highly-alloyed steels, or

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when the slab is rolled into thin plates [2]. During solidification process, continuously cast steel undergoes volume shrinkage or deformation while transitioning from liquid phase to solid phase, as shown in Figure 2. The gap between supporting rolls, which defines the taper of the entire casting machine, is designed to decrease along the caster to compensate for the volume changes. Strand centerline is susceptible to segregation and other defects if proper compensation is not achieved. Soft reduction technology has been developed to reduce centerline segregation and related centerline defects. The caster from Figure 1 usually has rollers with small radius, whose location can be adjusted in real time to impose a reasonable reduction profile to compensate for liquid core shrinkage. The optimum point to apply reduction is the solidification end (the ML). If the reduction is applied after the solidification end, then the steel is completely solid upon entering the region, and the large forces between the rolls and the strand damage both. If the reduction is applied before the solidification end, then the soft reduction is insufficient where it is needed and centerline defects will arise. Therefore, the soft reduction operation performs best when the ML stays constant with time.

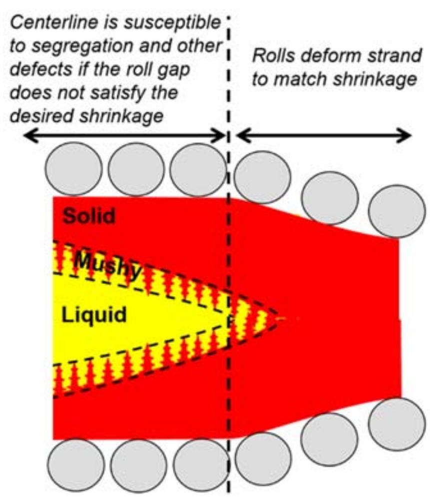


Figure 2. Soft reduction operation to reduce centerline segregation.

The potential of using open-loop control for the task of maintaining ML during casting speed drops was explored in [3], and the results show that bang-bang control has the best performance among the methods studied. However, the well-posedness and rigorous mathematical formulation of bang-bang control as applied to steel casting have not been addressed, and the study was limited to the casting speed drop.

The continuous steel casting process has natural nonlinear infinite-dimensional representations in terms of the Stefan problem partial differential equation (PDE) [4] and the more detailed enthalpy formulations [5]. These are simplified 1D models which could be assembled into a 2D cross-section transient model of the three-dimensional (3D) slab solidification process through spatial step interpolation [6]. Measurements are only available at particular points in the caster, each corresponding to a single discrete-in-time boundary point [7]. Two models are considered in this paper. The first model is based on the 1D Stefan problem, using continuous boundary measurements. The second model is an industrial grade 2D transient process model based on the 1D enthalpy formulation [3]. Existence and uniqueness of solutions to the various Stefan problem, in contrast to the enthalpy method, is very well studied [4] [8] [9]. The two-phase Stefan problem with Dirichlet and Neumann boundary conditions imposed at both liquid and solid boundaries under minimum smoothness assumptions on data was studied, respectively, in [10] and [11]. Results on Stefan problem relevant to this work are also given in [12]-[17].

In the present work, first, the existence and uniqueness of the solution of the single-phase 1D Stefan model [7] under bounded bang-bang control and some simplifying but practically justified assumptions are proved. Then, the result is extended to the 2D interpolation-based model. Next, the general synthesis setting for bang-bang control of the 1D single-phase Stefan problem and its interpolation-based 2D extension under boundary flux inputs with hard constraints is formulated. Then, the bang-bang control for the minimization of the ML deviation from the desired value after the casting speed increase is heuristically found for the 2D slab model through trial-and-error, and the resulting simulation is provided.

The proofs of formal statements are omitted due to space limitation and will be provided in a separated publication.

II. Mathematical models

In solidifying steel, heat is transferred by diffusion and advection. However, through a scaling argument, diffusion heat transfer can be neglected, as the advection dominates the conduction in the casting direction. By taking reference frame that moves with the material down through the caster, the entire 3D problem can be modeled as a 2D slice of the material with reasonable accuracy. Furthermore, in slab casters, heat transfer along width direction is negligible. Therefore, a 1D transverse slice with through-thickness solidification at the center reasonably well represents the entire 2D slice solidification and gives good accuracy, except, possibly at the corners, where the model needs to be slightly adjusted. The 1D slice will be used for this work. To simplify the notation, and to permits generalization of the results to the entire strand, the work in this paper will also assume the temperature is symmetric across the center of the strand.

A. Single-Phase Stefan Problem

The domain of the moving 1D slice of the caster is divided into two separate sub-domains, solid steel and liquid steel. Within each subdomain, temperature evolves according to the usual linear parabolic heat diffusion equation. The liquid-solid boundary moves according to the conservation of energy between the heat fluxes - proportional to the temperature gradients on either side of the boundary and the latent heat of solidification.

Let x be the spatial variable, t be the time variable, $T(x,t)$ be the temperature, L be the half-thickness of the slab, and $s(t) \le L$ be the location of the liquid-solid interface. Then, the Stefan Problem is written as:

$$T_t(x,t) = aT_{xx}(x,t), \quad x \in (0,s(t)) \cup (s(t),L), \tag{1}$$

$$T(s(t),t) = T_f, \tag{2}$$

$$T_x(0,t) = u(t), \quad T_x(L,t) = 0, \tag{3}$$

$$T(x,0) = T_0(x), \quad s(0) = s_0, \tag{4}$$

$$s_t(t) = b\left(T_x(s^-(t),t) - T_x(s^+(t),t)\right), \tag{5}$$

where the material is solid for $x \in (0, s(t))$ and liquid for $x \in (s(t), L)$, T_f is the melting temperature, a is the thermal diffusivity, $k/\rho c_p$, $b = k/\rho L_f$, the properties: k (thermal conductivity), ρ (density), c_p (specific heat), and L_f (latent heat) can vary with temperature, but stay strictly positive ([5])

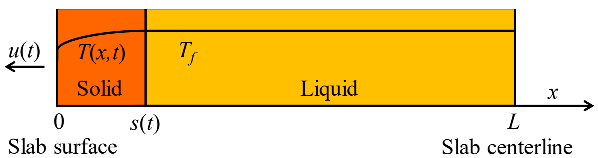


Figure 3. Simulation domain of 1-D slice through slab thickness.

and are assumed to be positive constants in this paper. $u(t)$ is the boundary heat flux. In the equations above, subscripts x and t indicate partial derivatives.

To simplify the problem, several assumptions were made.

(A1) $T_0(x)$ is continuous, $T_0(x) \le T_f, x \in (0, s_0)$ and $T_0(x) = T_f, x \in (s_0, L)$.

That is, the steel is initially below the melting temperature in the solid and equal to the melting temperature in the liquid. The first condition is physically necessary. The second condition is a reasonable simplification. The temperature of the liquid steel entering the mold from the tundish is higher than the melting temperature, and the difference is called superheat. Most of this superheat is removed by the time the molten steel reaches the meniscus, where the solidification begins. The superheat in the liquid in a caster is around 25 to 50 °C, while the temperature at the strand surface is hundreds of degrees below the melting temperature. Therefore, temperature gradients in the liquid pool can be ignored. The initial temperature $T_0(x)$ is the temperature at the meniscus. Since the superheat entering the mold is usually monitored online at steel plants, the temperature at the meniscus can be treated with very small error as uniform and equal to the pouring temperature. Under this assumption, the slice domain is shown in Figure 3.

(A2) $s_0 > 0$.

In equation (4), the initial temperature distribution across (0, L) is assumed known and uniformly equal to the pouring temperature, which leads to $s_0 = 0$. However, in the real process, sometimes a little superheat remains in the liquid when it first touches the mold wall, so solidification starts slightly below the top of the liquid level in the mold ($s_0 = 0$). For other situations, the meniscus can solidify slightly so that the shell thickness at the top of the liquid level, is already slightly positive ($s_0 > 0$). In both cases, s_0 is reasonable approximations usable in a simple model.

(A3) $u(t)$ is a bounded piecewise continuous non-negative function.

This assumption is physically justified since in the actual caster, the feasible range of the cooling water flow rates is hard-constrained by the physical limitation of the spray cooling system, so the possible flux at the surface is hard-constrained as well, and piecewise continuous. The sign of $u(t)$ means that heat is leaving the region.

The existence and uniqueness of the solution to the Stefan problem (1)-(5) under assumptions (A1)-(A3) will be addressed in Subsections IIIA and IIIB.

From [7], assumptions (A1) and (A3) jointly imply the second useful consequence:

$$T < T_f, \quad x \in (0, s(t)); \quad T = T_f, \quad x \in (s(t), L). \tag{6}$$

Then, the Stefan condition (5) simplifies to:

$$s_t(t) = bT_x(s^-(t),t). \tag{7}$$

1) Existence

Lemma 1: The solution $T(x,t)$ of problem (1)-(4) under assumptions (A1)-(A3) for a real-valued function $s(t)$ satisfies:

$$0 \le T_x(s^-(t),t) \le M \tag{8}$$

where $M = \max\{\max_{t,x}\{u(t), \frac{T_f - T_0(x)}{s_0 - x}\}\}$.

Lemma 2: Under assumptions (A1)-(A3), there exists a solution of problem (1)-(5) that is defined for all $t > 0$.

2) Stability, Uniqueness, and Monotone Dependence

Let (T_i, s_i), i=1,2, denote solutions of the Stefan problem (1) - (5) for the respective boundary and initial conditions u_i, T_0^i and s_0^i, i=1,2, which satisfy the assumptions (A1)-(A3). Suppose that $s_0^1 \le s_0^2$. Then the following results are valid.

Lemma 3: Under the assumptions (A1)-(A3), the solution to the Stefan problem (1)-(5) is unique.

B. Enthalpy PDE

Enthalpy PDE, which is a generalized form of conservation of energy, is used in [5] to model the 1D slice:

$$H_t(T(x,t)) = \left(k(T(x,t))T_x(x,t)\right)_x, \quad x \in (0,L) \tag{9}$$

where $H(T)$ calculates the enthalpy—the thermodynamic internal energy—of the material at temperature T, and $k(T)$ is the temperature-dependent conductivity. The boundary conditions remain the same as in (3), and assumption (A1) can be similarly stated to ensure the problem is physically realistic.

It can be shown [15] that (9) and (1), (2), and (5) are equivalent in a weak sense. Suppose k is constant, and H is defined as

$$H(T(x,t)) = \rho\left(c_p + L_f \bullet 1(T(x,t) - T_f)\right) \tag{10}$$

where ρ and c_p are the constant density and specific heat respectively, and $1(\cdot)$ is the unit step function. Then the weak forms of (9)-(10) and (1), (2), and (5) under the assumption (A1) are the same.

While (9) is difficult to analyze mathematically, it may be numerically modeled on a fixed computational domain, with (10) slightly mollified, giving a similar function $H(T)^*$ that is differentiable. The effective specific heat can then be defined as the derivative of enthalpy,

$$c_p^*\left(T\left(x,t\right)\right)=\frac{dH^*\left(T\right)}{dT} \quad . \tag{11}$$

Then, applying the chain rule, equation (8) becomes:

$$\rho c_p^*\left(T\left(x,t\right)\right)T_t\left(x,t\right)=\left(k\left(T\left(x,t\right)\right)T_x\left(x,t\right)\right)_x \tag{12}$$

Although equation (12) has the form of a simple transient heat conduction problem, the effective specific heat c_p^* is nonlinear, being much larger during phase changes than at other temperatures. Using (11)-(12), an effective specific heat based model is employed in [5] for a single slice model, and in [6] for a multi-slice model. The latter is referred to in [6] as CONSENSOR, which represents an industrial-grade software sensor, formulated as the 2D temperature/(shell thickness) state estimator that is open-loop in the spray-cooling zone but has closed-loop initial condition adjustment at the mold exit based on the real-time inlet/outlet mold water temperature reading. This approach permits a better matching of process data in modeling alloys [6], since their solidification occurs over a range of temperatures and positions, rather than at a single point. CONSENSOR, currently used in production, is employed in Section VI to demonstrate the applicability of the results derived on the basis of (1)-(5) to the actual industrial process.

III. Control Problem Formulation

Profitability changes according to many circumstances that introduce significant complexity in determining optimal casting conditions. For the steel grade studied in [3] the concerns about centerline segregation dominate. For the casters that employ soft reduction technique to mitigate centerline segregation, the ML control is of central significance.

The present paper considers a problem complementary to the reduction of the sudden-speed-drop-induced ML deviation considered in [3]: minimizing the ML deviation during a sudden, *and more dangerous due to the possibility of the molten steel escape*, casting speed increase from v_1 to v_2 in a thick-slab caster, completing thereby a preliminary study of the bang-bang control application to ML deviation reduction under sudden speed changes in continuous steel casting.

A. Multi-slice Model

As indicated above, enthalpy PDE (9) and Stefan problem (1), (2), and (5) are equivalent in a weak sense. CON1D is a single-slice model [5] employing (11)-(12), which represent a generalized version of (9). The existence and uniqueness of the solution of the single-slice Stefan problem have been shown in Section IIA. The same conclusion also follows for the equivalent single-slice model, CON1D.

As shown in Figure 1, CONSENSOR [6] repeats the CON1D calculation for multiple slices simultaneously, producing the temperature profile along the entire caster (z) and through its thickness (x) in real time. CONSENSOR uses delay interpolation method [6] to search for the latest temperature histories available from each CON1D slice at locations along the caster.

Theorem 1: Let $T_i(x, t)$ be the solution to i^{th} slice of the single-phase Stefan problem (1)-(5) under the assumption (A1)-(A3), let z be the casting direction. Then the solution $T(x,z,t)$ of the CONSENSOR model exists and is unique.

B. Single-slice Based Optimization Problem

A schematic of different slices in a part of the caster is shown in Figure 4. Suppose the ML is z_{ml} while casting at v_1. When the casting speed suddenly increases to v_2 at $t = 0$, slices are in various locations in the caster with various shell thickness and temperature distribution across x-axis. Take slice i in Figure 4 as an example. When casting speed increases ($t = 0$), it is at location z_i with shell thickness s_0^i and temperature distribution $T_0^i(x)$.

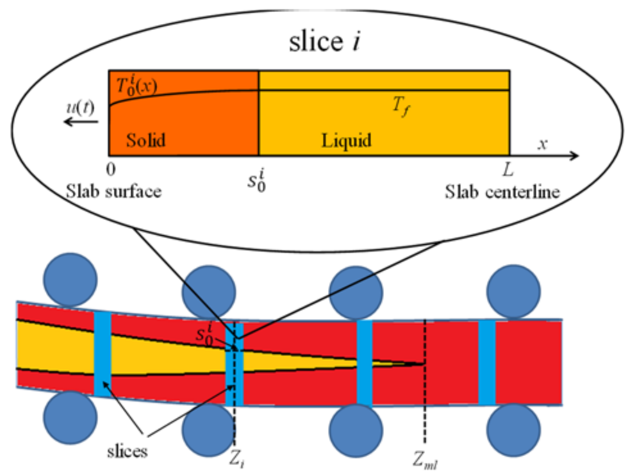


Figure 4. Illustration of problem formulation for different slices in the caster during the speed increase.

Let t_1^i and t_2^i be the times when the slice i reaches z_{ml} under casting speeds v_1 and v_2 respectively. Then we have

$$t_1^i=\frac{z_{ml}-z_i}{v_1},\quad t_2^i=\frac{z_{ml}-z_i}{v_2}, \tag{13}$$

and since $v_1 < v_2$, we have $t_1^i > t_2^i$. After the speed increase, if $u(t)$ stays the same, then for slice i, $s^i\left(t_1^i\right) = L > s^i\left(t_2^i\right)$, where L is the half slab thickness. This means that the new ML at v_2 will be greater than z_{ml}, which is consistent with the simulation result in Figure 5a.

The objective is to minimize the ML deviation. The ML is the location where the slice is fully solid, i.e. $s^i(t) = L$. If $s^i\left(t_2^i\right) = L$ and $s^i(t) < L, 0 < t < t_2^i$, the ML of slice i at casting speed of v_2 is also z_{ml} and the ML deviation is 0. Therefore, the objective is modified to the following: to minimize the difference between $s^i\left(t_2^i\right)$ and L.

Define the cost function

$$J^i=\left|s^i\left(t_2^i\right)-L\right| \tag{14}$$

with a control function $u(t)$ set

$$K=\left\{N_1\le u\left(t\right)\le N_2,\quad N_1,N_2>0\right\}.$$

The hard constraint N_1 stands for the fact that the cooling sprays are never completely shut off for technical reasons so that the water flow rate has a lower bound. The hard constraint

N_2 stands for the fact that the spray cooling servo has spray rate limit.

Problem (I). Find $u_o^i \in K$ such that

$$J^i\left(u_o^i\right) = \min_{u \in K} J\left(u\right).$$

Results in [3] show that under sudden speed drop, the *undershoot* right after the speed change in ML profile is unavoidable. Simulation result in Figure 5b shows that under sudden speed increases, an *overshoot* in ML profile is also unavoidable. Therefore, for slices close to z_{ml} at $t = 0$, $s^i(t_2^i) < L$. The following optimization problem is considered first.

Problem (II). Find $u_o^i \in K$ such that

$$s_{u_o^i}^i\left(t_2^i\right) = \max_{u \in K} s_u^i\left(t_2^i\right)$$

where $s_u^i(t)$ represents the solution of free boundary to the Stefan problem (1)-(5) under initial condition $T_0^i(x)$, s_0^i and boundary condition $u(t)$ for slice i.

Theorem 2: There exists a solution of Problem (II).

Theorem 3: There exists a solution u of Problem (II), given by:

$$u\left(t\right) = N_2, \quad t < t_2^i. \tag{15}$$

C. Multi-slices Based Optimization Problem

The discussion in Subsection IIIB focuses on a single slice, but CONSENSOR [6] manages N (from 200 to 400) slices with a uniform spatial interval in between. When the casting speed changes, the maximum ML deviation during the transition is determined by the slice that has the largest ML deviation among all slices. As shown in Figure 4, N slices simulated by CONSENSOR are at various locations in the caster. For each slice, its shell thickness and temperature distribution at $t = 0$ are treated as new initial conditions, denoted by s_0^i, $T_0^i, i = 1,2,..,N$. Since slices are at various locations, they have different dwell time to travel from their location to z_{ml}, denoted as $t_2^i, i = 1,2,..,N$.

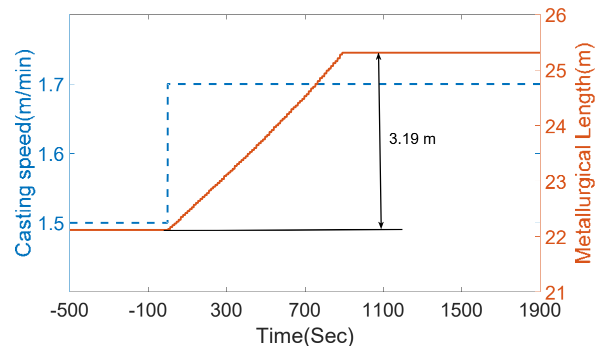
For each slice, Problem (I) will be solved. And the slice that has

$$J = \max_{i=1,2,\ldots,N} J^i\left(u_o^i\right) \tag{16}$$

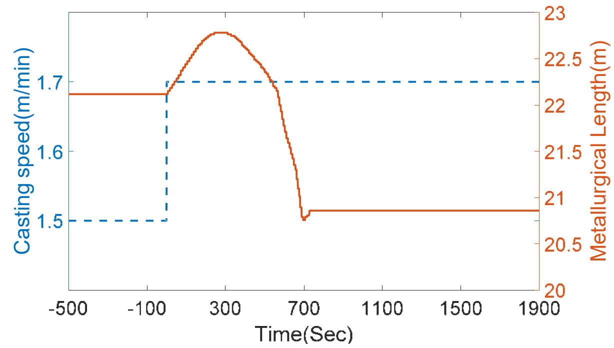
determines the maximum ML deviation.

However, in the real continuous casting process, there is another significant constraint, currently not reflected in the formal optimization problem setting: the spray-cooling region is divided into several aggregated spray zones, each linking a group of spray nozzles together. A spray zone typically can only have just one spray flow rate over the entire zone. Therefore, the control variable $u(t)$ is not independent of different slices: when we determine u_0^i that solves problem (I) for slice i, $u^{i+1}(t)$ has also to be determined for slice i+1 for the time that these two slices stay in the same spray cooling zone.

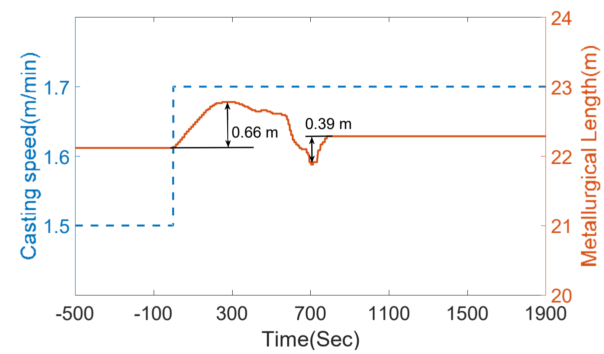
At this time solution to the above optimization problem is not yet formally derived and will be part of future work. The control signal in Selection IV has been computed heuristically through simulations, *considering spray zone aggregation*.



(a) No control.



(b) Single speed change to maximum flow rate after speed increase.



(c) Two-step bang-bang control sequence

Figure 5. Model prediction of metallurgical length during speed increase under different control methods.

IV. Simulation and Discussion

The simulation is carried out based on the model of the thick-slab (221 mm) caster at JFE Steel, Japan [3] under casting speed increase from 1.5 m/min to 1.7 m/min. The simulated steel properties, casting conditions, and physical limitation of the spray cooling system are those given in Tables 1, 2, and 3 of [3]. The parameters k, ρ, L_f, and c_p vary with temperature T according to [2]. Theorems 2 and 3 show that to minimize the deviation of shell thickness after speed increase, the boundary cooling flux, i.e. the spray flow rate, needs to be maximized. Due to hard constraints of spray, bang-bang control method has been considered.

Two-step bang-bang control sequence has been developed heuristically by finding boundary heat flux commands to minimize the deviation of average shell thickness at each spray zone sequentially through simulations in order to minimize the ML deviation. The following two-step bang-bang control sequence was proposed:

$$u_{2b}^{i}(t)=\begin{cases} u^{i}(orig)\ if\ \ t<0, \\ u^{i}(max)\ if\ \ 0<t<t_{2b}^{i}, \\ u^{i}(final)\ if\ \ t\geq t_{2b}^{i}, \end{cases} \tag{17}$$

where u_{2b}^{i} is the boundary heat flux, t_{2b}^{i} is the switching time for spray zone i, $u^{i}(orig)$, $u^{i}(max)$, and $u^{i}(final)$ are the original, maximum, and final heat fluxes at spray zone i. $u^{i}(orig)$ and $u^{i}(final)$ are hypothetical, but still realistic, which achieves the same ML at casting speeds of 1.5 m/min and 1.7 m/min given in Table 4 of [3].

The resulting ML profile under the two-step bang-bang control sequence is shown in Figure 5c. The maximum ML deviation is 0.66 *m*. Compared with the constant spray cooling case in Figure 5a, the ML deviation is reduced by 79.4%. It appears that the ML control during speed increase is more effective than during speed drop (with minimum ML deviation of 0.66m for speed increase compared with 0.8 m [3] for speed drop). However, the performance is limited by hard spray flow rate constraints.

The results show that the practical ability to control ML variations in this thick-slab caster is relatively limited due to hard constraints on the water flow rates. However, in the thin-slab and the medium-slab casters, ML variations are expected to be controlled much more effectively.

V. Conclusion and Future Work

In this paper, the existence and uniqueness of the solution to Stefan problem under bang-bang optimal control (piecewise continuous and bounded) signal are proven. An ML control problem in continuous casting of steel is formulated and solved numerically. Finding the analytical solution to the proposed optimization problem will be addressed in future work.

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