Enthalpy-based Full-State Feedback Control of the Stefan Problem with Hysteresis*

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***Abstract*—This paper presents a full–state controller design with respect to a reference solution for the one-phase Stefan problem under input hysteresis. The setting models an industrial casting processes with water cooling hysteresis under Neumann boundary actuation. The control law proposed ensures exponential stability of average enthalpy and is proven to provide asymptotic convergence of temperature error and solidification front error. A simulation supports the result.**

I. INTRODUCTION

Consider a continuous steel casting solidification process. The latter is typically modeled by the Stefan problem [1] with moving boundary between liquid and solid phases. Transverse surface cracks may be created when the curved strand of steel undergoes unbending, which causes tensile stress on the inside radius surface, if the steel surface is too brittle. Since ductility of steel is strongly temperature-dependent, these cracks can be avoided by regulating the surface temperature outside of the embrittlement temperature range. The creation of internal cracks, usually found below the strand surface, depends on the relative location of the solidification front down the caster. Another defect, strand bulging past support rolls, called a ”whale”, damages the casting machine and causes a work stoppage. Hence, regulating the entire distributed temperature profile and solidification front of the steel [2] is required.

The current industry-standard control method is open-loop control that changes the cooling water spray flow rates according to the casting speed and given spray patterns. The latter define the flowrates in each spray zone for each casting speed in the table, which depends on the steel grade, product dimensions, and machine design. The spray pattern can be determined by experience, or through offline optimization techniques [3]–[6]. In the control literature on the Stefan problem, there have been many techniques suggested. The approach used in [7] is to solve the inverse Stefan problem, i.e. impose a desired trajectory of the boundary and determine computationally a temperature profile. This would satisfy the whale constraints but could still result in temperature related cracks. In [8], a full-state feedback control law for a single-phase Stefan problem is designed by introducing a nonlinear backstepping transformation. Based on this technique, [9] designed an observer-based output feedback control law that achieved exponentially stability of the moving interface to a fixed reference setpoint. However, the convergence of a moving boundary to a fixed solid/liquid interface is not enough for solidification process like the continuous casting of steel, since neither whale constraints nor cracks constraints are satisfied. These constraints require convergence to the moving interface profile. In [10], the authors control the position of the solidification front, neglecting surface cracks using thermostat-style boundary control inputs. In [11], the authors found a control law that ensured convergence of both temperature and solidification front position to the desired reference profile for a one-phase Stefan problem with practical assumptions on the actual casting process. No previous work has considered the phenomenon of hysteresis.

In this paper, a free boundary problem with hysteresis type boundary conditions is considered. Hysteresis effects, which can be characterized as a special type of memory-based relation between an input signal $v(t)$ and an output signal $w(t)$, arises in different areas of science, such as physics, engineering, economics, and biology. There is an extensive body of research concerning the modeling of hysteresis [12], [13]. A generic approach to controlling hysteretic systems is to combine inverse compensation with feedback [14]–[16]. There are limited studies on control of Stefan problem with hysteresis. [10] considered a two-phase Stefan problem with hysteresis for a simple situation of thermostat control. Friedman [17] considered optimal control of the free boundary of a two-phase Stefan problem with hysteresis-type boundary conditions. Periodic control of a two-phase Stefan problem with Dirichlet boundary control with hysteresis was considered in [18]. However, all these previous works propose hysteresis effects on the free boundary, rather than at the surface where heat flux is applied.

The existence of hysteresis [19] in a certain temperature interval, points to significance of considering the thermal history of actual cooling processes when designing controllers, which is the subject of the present work. This paper is organized as follows. The process model is presented in Section II, and the hysteresis model is given in Section III. The control objective and our main result, a control law that guarantees simultaneous asymptotic convergence of the temperature and solidification front location to the reference profile, are stated in Section IV. Supporting numerical simulations are provided in Section V. In Section VI, some extensions of this framework to observer and output feedback control are briefly discussed.

II. PROCESS MODELS

A. Single-phase Stefan Problem

The continuous steel casting process can be modeled accurately using a one-dimensional spatial domain of a moving 1-

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D slice divided into two separate sub-domains, corresponding to the solid and the liquid steel phases [3]. The material is solid for $x \in (0,\ s(t))$ and liquid for $x \in (s(t),\ L)$, L is the half-thickness of the slab. The heat flux removed from the material at the solid surface is directly related to the controlled flow rate of the water spray applied at the solid surface. The relation is a subject of ongoing research in the steel industry, but Nozaki's relation [20] has been successfully used by many modelers [2]–[4]. The following practical assumption, discussed in detail in [21], is made:

(A1) The initial conditions satisfy: $0 < s_0 < L$, $T_0(x) < T_f$, is continuous and non-decreasing for all $0 \leq x < s_0$, and $T_0(x) = T_f$ for all $s_0 \leq x \leq L$.

With (A1), the following PDE models the evolution of temperature within the slice

$$T_t(x,t) = aT_{xx}(x,t),\ \ x \in (0, s(t)), t \in \left(0, t_f\right) \tag{1}$$

$$T\left(s(t), t\right) = T_f,\ \ t \in \left(0, t_f\right),\ \ (s(0) = s_0 > 0) \tag{2}$$

$$T_x(0,t) = u(t), t \in \left(0, t_f\right) \tag{3}$$

$$T(x,0) = T_0(x),\ \ x \in (0, L), \tag{4}$$

$$\dot{s}(t) = bT_x(s^-(t), t), t \in \left(0, t_f\right) \tag{5}$$

where $a = k/\rho c_p$, $b = k/\rho L_f$, T_f is the melting temperature, k is the thermal conductivity, ρ is the density, c_p is the specific heat, L_f is the latent heat, and $u(t)$ is the boundary heat flux.

B. Enthalpy Formulation

The thermodynamic energy of the material is called enthalpy. In a single-phase material, the enthalpy is approximately proportional to the temperature, with the constant of proportionality equal to c_p. However, for a solidifying pure material, there is a step change in enthalpy at the melting temperature, T_f, equal to the latent heat. Altogether, this means the enthalpy, denoted as h, can be described as a function of temperature:

$$h(T) = \begin{cases} c_p T & T < T_f \\ c_p T + L_f & T > T_f \end{cases} \tag{6}$$

In control theory terminology, enthalpy is the actual state variable of the system. The following PDE is physically equivalent to the Stefan Problem:

$$\rho h_t\left(T\left(x,t\right)\right) = kT_{xx}\left(x,t\right),\ x \in \left(0, L\right), t \in \left(0, t_f\right) \tag{7}$$

$$T_x\left(0,t\right) = u\left(t\right),\ \ T_x\left(L,t\right) = 0,\quad t \in \left(0, t_f\right) \tag{8}$$

$$T\left(x,0\right) = T_0\left(x\right),\ \ x \in \left(0, L\right) \tag{9}$$

The weak forms of the two PDEs are the same, as discussed in [22]. Although it is much easier to write down and simulate numerically, this PDE is more difficult to analyze mathematically due to the discontinuity in the function $h(T)$. The enthalpy PDE is used in this paper for the simulation.

III. Hysteresis Model

A. Hysteresis due to boiling

The heat extraction by cooling water is governed by water boiling phenomena, which greatly depend on the temperature. In the spray cooling region of the continuous caster, cooling water droplets impinge onto the hot steel surface, and vaporize immediately to form a stable steam layer. The latter prevents subsequent water droplets from coming in contact with the steel strand surface and decreases the heat removal.

An experiment was done to determine the boiling curve from 200 ^oC to 1200 ^oC and then back to 200 ^oC for a Pt-specimen under air-mist spray nozzles [19]. The curves for both temperature processing histories (Figure 10 in [19]) reveal hysteresis due to boiling in the water cooling process, as time is needed for the steam layer conditions to change, and this hysteresis depends on the surface temperature of the strand. Let $H(\cdot)$ denote the hysteresis operator, then:

$$u(t) \quad = \quad H(T(0,t), q(t)) \tag{10}$$

where $q(t)$ denotes the accessible control input, i.e., the heat extraction due to the spray cooling water. The configuration of the process model is shown in Figure 1.

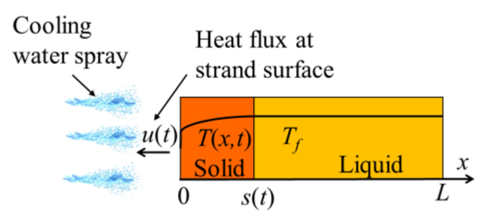


Fig. 1: Schematic of 1D Stefan problem with hysteresis

In an actual caster, the feasible range of the cooling water flow rates is hard-constrained by the physical limitation of the spray cooling system. Therefore, the following assumption is made:

(A2) $q(t)$ is a bounded piecewise continuous function, i.e.$M_1 \leq q(t) \leq M_2, M_1, M_2 > 0$.

A simple way to account for boiling heat transfer effect that has been successfully used by other modelers [23] is to introduce a heat flux multiplier into (10),

$$u\left(t\right) = H(T(0,t), q(t)) \quad = \quad F\left(T(0,t)\right)\ q(t), \tag{11}$$

where $F(T(0,t))$ is a simple hysteresis functional, as shown in Figure 2.

Mathematically, it can be represented as follows:

$$F(T(0,t)) = \begin{cases} 1 & \text{if } T(0,t) < T_\alpha \text{ or } T(0,t) > T_\beta \\ f_L(T(0,t)) & \text{if } \exists t^* \in (0,t], \text{s.t. } T(0,t^*) \geq T_\beta \\ & \quad \text{and } T(0,\tau) > T_\alpha \forall \tau \in [t^*, t] \\ f_H(T(0,t)) & \text{if } \exists t^* \in (0,t], \text{s.t. } T(0,t^*) \leq T_\alpha \\ & \quad \text{and } T(0,\tau) < T_\beta \forall \tau \in [t^*, t] \end{cases} \tag{12}$$

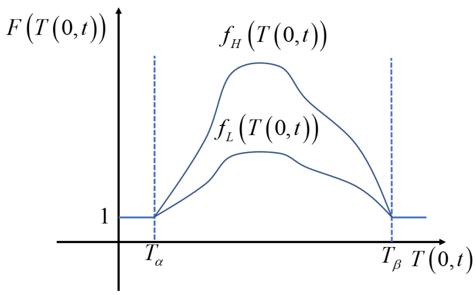


Fig. 2: Hysteresis loop

where $f_L(\cdot)$ and $f_H(\cdot)$ are the heat flux multipliers that can be measured in the laboratory through experiments; another assumption is made:

(A3) $f_L(\cdot)$ and $f_H(\cdot)$ are known continuous functions.

B. Existence of solution

Theorem 1: Under assumptions(A1)–(A3), there exists a unique solution to (1)-(5) with boundary condition (11), which is defined for all $t > 0$.

Proof: Since typically, $T_0(0) > T_\beta$, without loss of generality assume $F(T_0(0)) = 1$. Take any small $\delta > 0$ and define:

$$F^\delta(t) \equiv 1, \text{ if } 0 < t < \delta. \tag{13}$$

Based on Theorem 1 and Theorem 2 in [6], a unique solution to the system (1)-(5) with boundary condition (11) exists. Denote the solution as (T^δ, s^δ). Next, define

$$F^\delta(t) = F(T(0,t-\delta)), \text{ for } \delta < t < 2\delta. \tag{14}$$

Since $F^\delta(t)$ is known for this interval, we can solve (1)-(5) for $\delta < t < 2\delta$ with this $F^\delta(t)$ and with the initial values $(T^\delta(x,\delta), s^\delta(t))$.

Now define $F^\delta(t)$ for $2\delta < t < 3\delta$ as above in terms of the function $T(0,t)$ obtained in the second step, and solve (1)-(5) for $2\delta < t < 3\delta$. Continuing in this way we obtain a unique solution of (1)-(5) with boundary condition (11) with $F(t) = F^\delta(t)$.

From Lemma 1 in [6], there exists a constant $C > 0$ such that

$$0 \le \frac{d}{dx} T^\delta(s^{\delta -}, t) \le C$$

i.e.

$$0 \le \dot{s}^\delta(t) \le C. \tag{15}$$

From (15), the functions s_δ form an equicontinuous, uniformly bounded family. Hence Ascoli-Arzela's theorem holds and we can select a subsequence s_δ that converges uniformly to $s(t)$ as $\delta \to 0$, and T^δconverges uniformly to T, i.e. (T,s) will be a solution. ■

IV. Controller Design

A. Control objective

Assume that a known reference temperature $\bar{T}(x,t)$ and solidification front position $\bar{s}(t)$ have been determined that are the solutions to (1)-(5) under known reference control input $\bar{q}(t)$ with initial conditions $\bar{T}(x,0) = \overline{T}_0$ and $\bar{s}(0) = \bar{s}_0$. This reference temperature profile should satisfy the quality goals and constraints of the process, and could, for example, be calculated for the continuous caster via the optimization methods [5], [6]. That is, matching the reference temperature should result in the safe operation and good quality material. One more assumption on the reference profile is required: $\dot{\bar{s}}(t) \ge 0$ for all $t \ge 0$.

Denote the reference tracking errors as $\tilde{s}(t) = s(t) - \bar{s}(t)$, and$\tilde{T}(x,t) = T(x,t) - \bar{T}(x,t)$. Subtracting PDE (1) – (5) with the reference control and solution from that with control to be determined yields,

$$\tilde{T}_t(x,t) = a\tilde{T}_{xx}(x,t), \quad x \in (0,L) - \{s, \bar{s}\}. \tag{16}$$

$$\tilde{T}_x(0,t) = F(T_s(t))q(t) - F(\overline{T_s}(t))\bar{q}(t), \tag{17}$$

$$\tilde{T}_x(L,t) = 0. \tag{18}$$

Also, since solutions to (1) are twice spatially differentiable away from the solidification front, they must have continuous first spatial derivatives. Thus, if $\bar{s}(t) \ne s(t)$, then $\bar{T}_x(s^+(t),t) = \bar{T}_x(s^-(t),t)$, so

$$\dot{s}(t) = b\left(\tilde{T}_x\left(s^-(t),t\right) - \tilde{T}_x\left(s^+(t),t\right)\right), \tag{19}$$

$$\dot{\bar{s}}(t) = -b\left(\tilde{T}_x\left(\bar{s}^-(t),t\right) - \tilde{T}_x\left(\bar{s}^+(t),t\right)\right). \tag{20}$$

The control objective is to drive the moving interface $s(t)$ to the desired reference $\bar{s}$ while ensuring the convergence of $T(x,t)$ to $\bar{T}(x,t)$ as $t \to \infty$, by manipulating the heat flux $q(t)$.

Denote $s_1(t) = \min\{s(t), \bar{s}(t)\}$, $s_2(t) = \max\{s(t), \bar{s}(t)\}$. From here on, the notations will be simplified for clarity and space saving, using $T(x)$ to represent $T(x,t)$, or omitting both arguments altogether.

B. Preliminaries

Since (1) is parabolic on the sub-domains, if (A1) holds, then T_x is uniformly bounded (see Theorem 11.1, p. 211 of [24]) by a constant depending on the initial condition and bounds on q. Due to (5), this means that the solidification front speed is bounded, i.e.

$$0 \le v_{\min} \le \dot{s} \le v_{\max} < \infty. \tag{21}$$

Also $\left|\tilde{T}(0)\right|$ is bounded since

$$\left|\tilde{T}(0)\right| = \left|\tilde{T}(0) - \tilde{T}(L)\right| = \left|\int_0^L \tilde{T}_x dx\right| \le \sqrt{L}\left\|\tilde{T}_x\right\|_2. \tag{22}$$

As a consequence of Poincare's inequality:

$$\int_0^L T^2 dx \le 2T^2(s) + 4L^2 \int_0^L T_x^2 dx = 2T_f^2 + 4L^2 \int_0^L T_x^2 dx. \tag{23}$$

That is, both T and T_x are bounded in the $L_2(0,L)$ norm, and hence T is bounded in the Sobolev space $H^1(0,L)$. Similarly, Agmon's inequality ensures that $|T|$ is also uniformly bounded.

C. Control law

A generic approach to controlling hysteretic systems is to design hysteresis compensation, like inversion operator [14] to ensure accurate control. As shown in Figure 3, given a desired boundary heat flux u_d, and initial memory of hysteresis operator F, the inverse operator F^{-1} generates the required cooling water heat flux $q(t)$ to ensure $u = u_d$.

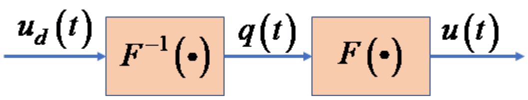


Fig. 3: Illustration of hysteresis inversion.

In [11], an enthalpy based control law is designed for system (1)-(5) without hysteresis. In the present work, hysteresis compensator is added to the control law to ensure controller performance for system (1)-(5) with boundary condition (11).

Define the "enthalpy function" $\eta(T)$:

$$\eta(T) = \begin{cases} \frac{1}{a}T, & \text{if } T < T_f, \\ \frac{1}{a}T + \frac{1}{b}, & \text{if } T \geq T_f. \end{cases}$$

The quantity η is proportional to the physical enthalpy h defined in (6). In this section, we will use the notation $\tilde{\eta} = \eta(T) - \eta(\bar{T})$ for the difference in enthalpy and

$$\tilde{H}(t) = \int_0^L \tilde{\eta}(t)dx = \frac{1}{a}\int_0^L \tilde{T}(t)dx - \frac{1}{b}\tilde{s}(t),$$

for the total difference.

Theorem 2: Suppose the initial conditions satisfy assumption (A1), the reference temperature profile and the reference spray water flux satisfy assumption (A2), and the boundary condition satisfies the control law:

$$q(t) = F^{-1}(T(0,t))F(\bar{T}(0,t))\bar{q}(t) + KF^{-1}(T(0,t))\tilde{H}(t) \tag{24}$$

where K is the controller gain. Then the reference temperature tracking error $\tilde{T}$ converges asymptotically to 0 uniformly over the domain, and the free boundary tracking error $\tilde{s}$ also converges to 0 asymptotically.

Proof: First, using equations (16)-(20),

$$\dot{\tilde{H}} = -(F(T_s(t))q(t) - F(\bar{T}_s(t))\bar{q}(t)).$$

Then, applying control law (24), yields

$$\dot{\tilde{H}} = -K\tilde{H}$$

which indicates $|\tilde{H}|$ decreases exponentially. As noted in Section IV.B, if all the assumptions are satisfied, T, $\bar{T}$ and consequently also $\tilde{T}$ are bounded in $H^1(0,L)$ over time. Then $|\tilde{s}|$ must also be bounded.

The proof is similar to the main proof given in [11]. The key to the proof is using an infinite-dimensional invariance principle from [25] to show convergence. Consider the Lyapunov functional candidate

$$V(\tilde{T}) = \frac{1}{2}\int_0^L \tilde{T}^2 dx - \frac{a}{b}T_f(s+\bar{s}) + 2\frac{a}{b}T_f L \tag{25}$$

on the state space of the error system, $(\tilde{T},\tilde{s}) \in H^1(0,L) \times \mathrm{R}$. This function is clearly continuous on that space, and non-negative on trajectories of the system.

Taking the time derivative of the first term of (25) along the trajectory of the system yields:

$$\begin{aligned} \tfrac{d}{dt}\tfrac{1}{2}\textstyle\int_0^L \tilde{T}^2 dx &= -a\tilde{T}(0)(F(T_s(t))q(t) - F(\bar{T}_s(t))\bar{q}(t)) \\ &\quad - a\textstyle\int_0^L \tilde{T}_x^2 \mathrm{d}x + \tfrac{a}{b}\tilde{T}(s)\dot{s} - \tfrac{a}{b}\tilde{T}(\bar{s})\dot{\bar{s}}. \end{aligned}$$

Also,

$$(F(T_s(t))q(t) - F(\bar{T}_s(t))\bar{q}(t)) = K\tilde{H} = K\tilde{H}(0)e^{-Kt}. \tag{26}$$

Differentiating the second term in (25) gives

$$\frac{d}{dt}\frac{a}{b}T_f(s+\bar{s}) = \frac{a}{b}T_f(\dot{s}+\dot{\bar{s}}). \tag{27}$$

Combining above equations yields

$$\begin{aligned} \dot{V}(\tilde{T}) &= -a\tilde{T}(0)K\tilde{H}(0)e^{-Kt} - a\textstyle\int_0^L \tilde{T}_x^2 dx \\ &\quad - \tfrac{a}{b}(\bar{T}(s)\dot{s} + T(\bar{s})\dot{\bar{s}}). \end{aligned} \tag{28}$$

From (28), the first term is exponentially decreasing. Under the assumptions, both $\dot{s}$ and $\dot{\bar{s}}$are positive and bounded below, as discussed above. Since the temperatures are bounded, choosing an appropriate temperature scale ensures that $\bar{T}(s)$ and $T(\bar{s})$ are also non-negative. So, after enough time,

$$\dot{V}(\tilde{T}) \leq -a\int_0^L \tilde{T}_x^2 \ .$$

Then, applying the Poincare inequality,

$$\begin{aligned} -a\textstyle\int_0^L \tilde{T}_x^2 dx &\leq -\tfrac{a}{4L^2}\textstyle\int_0^L \tilde{T}^2 dx + 2\tilde{T}^2(L) \\ &= -\tfrac{a}{4L^2}\textstyle\int_0^L \tilde{T}^2 dx = -W(\tilde{T}). \end{aligned} \tag{29}$$

Now, the invariance principle can be applied. Using the notation of [25], let $\mathscr{X}$ now be $(H^1(0,L) \times \mathrm{R}$, the state space of the problem, and $\mathscr{Y}$ be $C^0(0,L) \times \mathrm{R}$. Denote $\hat{W}$to be the extension of W to $\mathscr{Y}$. Then, $\mathscr{X}$ is compactly embedded in$\mathscr{Y}$, according to Rellich-Kondrachov theorem and the Ascoli-Arzela criterion [26]. The trajectories of the error system are bounded in $\mathscr{X}$. All conditions of the theorem are met, and therefore all trajectories of the system converge to the set

$$\mathscr{M}_3 \subset \{y \in \mathscr{Y} : \hat{W}(y) = 0\} = \{\tilde{T} \equiv 0\}$$

in the $\mathscr{Y}$-norm. That is, $\tilde{T}$ converges to 0 uniformly. Finally, since both $\tilde{T}$ and $|\tilde{H}|$ converge to 0, according to the definition of $\tilde{H}$, $\tilde{s}$ must converge to 0 as well. ■

V. Simulation

A. Numerical simulation

The performance of the proposed hysteresis compensated controller is investigated through numerical simulation by comparing to the performance of the following controller,

$$q(t) = \bar{q}(t) + K\tilde{H}(t) \tag{30}$$

which is the previous uncompensated controller design for the Stefan problem [11]. The specific initial conditions used are shown in Figure 4, the hysteresis functional used is represented by line segments shown in Figure 5, and the rest of the simulation parameters are given in Table I. The simulations employ an enthalpy-based method to model solidification, rather than an actual moving boundary. The simulation code was verified against an analytical solution to the Stefan problem from [27]. Varying $\bar{u}$ is used to simulate the spray zone, and the bounds of the control input are set to:

$$u_L = 0.01\bar{u}, u_H = 5\bar{u}. \tag{31}$$

TABLE I: Thermodynamic properties used in simulations.

Symbol	Description	Value
k	thermal conductivity	80.4 W/m · K
c_p	specific heat	460 J/kg · K
T_f	melting temperature	1538 ^oC
L	half-thickness of strand	0.002 m
ρ	density	7.87×10^3 kg/m^3

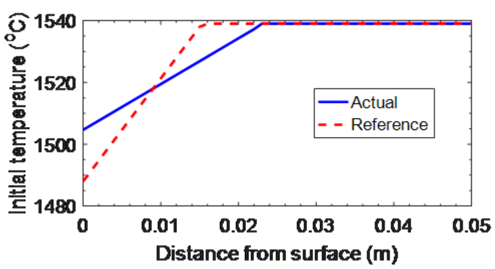


Fig. 4: Initial condition for simulations.

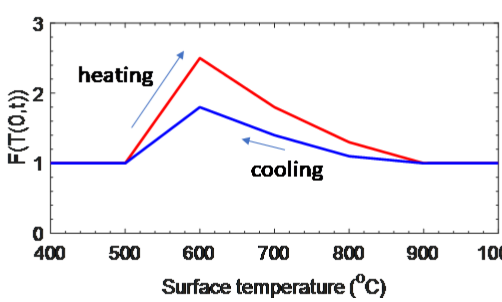
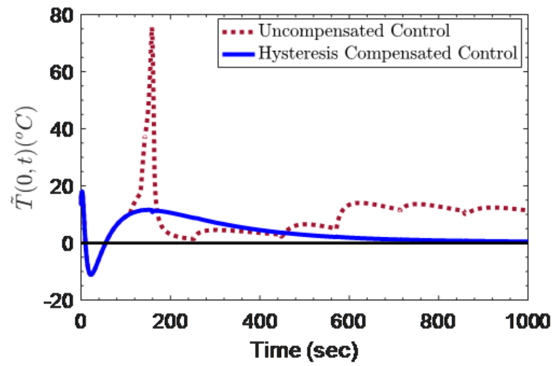


Fig. 5: Hysteresis loop for simulation.

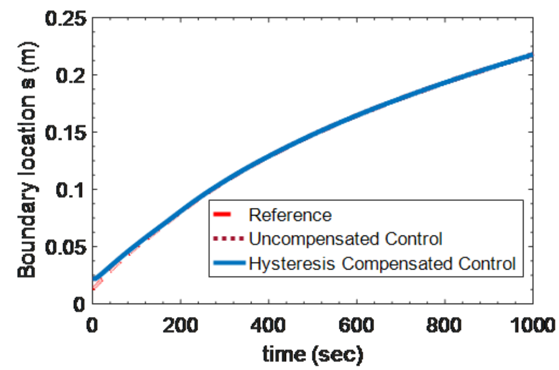
The behavior of the surface temperature reference tracking error $\tilde{T}(0,t)$, the closed-loop response of the moving boundary $s(t)$, and the manipulated heat flux, $q(t)$, are depicted in Figure 6. Hysteresis compensated control and uncompensated control show similar performance in driving $s(t)$ to $\bar{s}(t)$. However, for the surface temperature, compensated control ensures that $\tilde{T}(0,t) \to 0$ while the uncompensated control does not provide such a behavior. Due to the existence of the hysteresis, the compensated control varies more compared to the uncompensated control to account for the effects of the hysteresis.

VI. Discussion and Extension

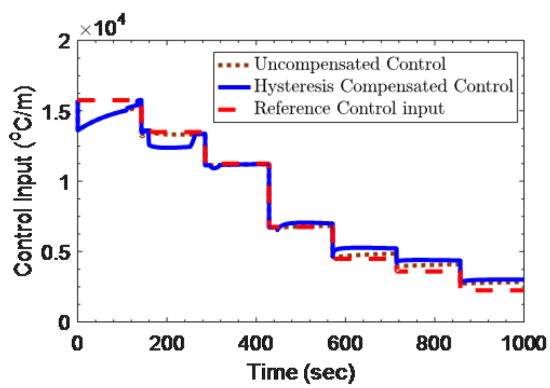
One weakness of the proposed control law lies in the assumption (A3) that the continuous hysteresis characteristics



(a) Temperature tracking error $\tilde{T}(0,t)$



(b) Solidification front $s(t)$



(c) Neumann boundary control $u(t)$

Fig. 6: Simulation of system (1)-(5) with initial condition mismatch control law (24).

can be determined based on lab experiments. However, the actual hysteresis characteristics may be different from the lab measurement or changing with time due to casting conditions changes. One approach is to design adaptive controllers with an adaptive hysteresis inverse, which will be a future work.

Another weakness lies in the full-state feedback control. Unfortunately, full-state feedback is not realistic for this problem. Enthalpy cannot be directly measured, only surface temperature,$T(0,t)$, can realistically be measured during casting. For an implementable control algorithm, an estimation scheme is needed. An estimator and output feedback control law has also been found which drive both estimation error and reference error to zero in simulation. The latter results will be presented in the subsequent publications due to space limitations.

VII. Conclusion

In this paper we presented a one-phase Stefan problem with hysteresis affecting the boundary flux input and proposed a hysteresis-compensating full-state feedback controller. The latter is proven and numerically shown to achieve asymptotic convergence to zero of the reference tracking errors for the temperature, unlike the uncompensated previously derived controller, and the solidification front position. An extension to the output feedback control will be provided elsewhere.

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